Spectral Graph Sparsification: overview of theory and practical methods

Yiannis Koutis

University of Puerto Rico - Rio Piedras

National Science Foundation
WHERE DISCOVERIES BEGIN

Graph Sparsification or 'Sketching'

Compute a smaller graph that preserves some **crucial** property of the input

Motivation: Computational efficiency with approximation guarantees

- **BFS**: Breadth First Spanning Tree
- **Spanner:** Spanning subgraph that approximately preserves distances

Spectral Graph Sparsification

Compute a smaller graph that preserves some **crucial** property of the input

We want to approximately preserve the eigenvalues and eigenvectors of the graph **Laplacian**

Motivation: Speed-up many clustering and partitioning algorithms based on computing Laplacian eigenvectors

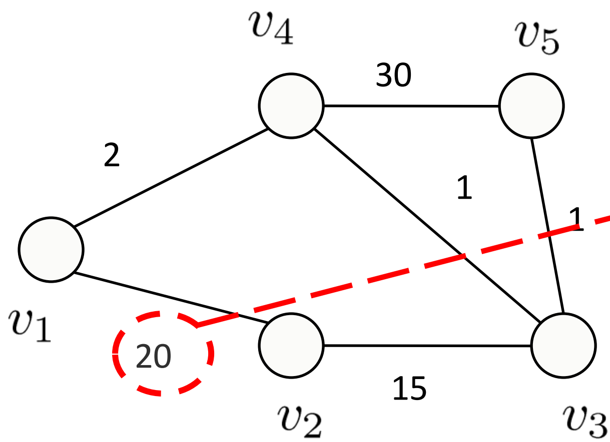
Spectral Graph Sparsification

Compute a smaller graph that preserves some **crucial** property of the input

We want to approximately preserve the **quadratic form** $\mathbf{x}^T\mathbf{L}\mathbf{x}$ of the Laplacian $\mathbf{L}$

Implies spectral approximations for both the Laplacian and the normalized Laplacian

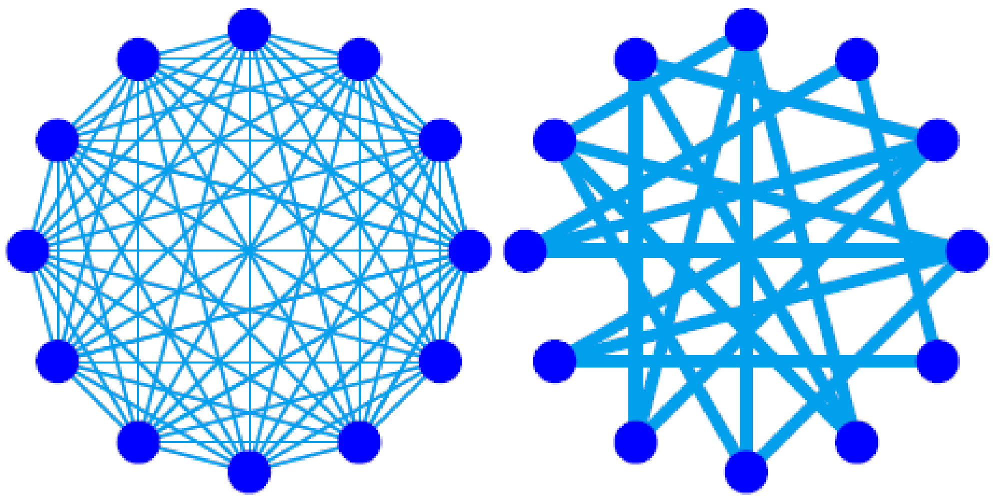
The Graph Laplacian



$$L = \begin{pmatrix} 22 & -20 & 0 & -2 & 0 \\ -20 & 35 & -15 & 0 & 0 \\ 0 & -15 & 17 & -1 & -1 \\ -2 & 0 & -1 & 33 & -30 \\ 0 & 0 & -1 & -30 & 31 \end{pmatrix}$$

Spectral Sparsification by Picture

$x^T L_G x$ $\quad$ $x^T L_H x$



- H is a reweighted subgraph of G
- H is obtained using randomness (sampling)

Combinatorial sketching

Spectral sketching

Linear system solvers

Better combinatorial sketching

Outline

- Sums of random positive matrices
- Combinatorial sketching to incremental sparsification
- Incremental sparsification for solving
- Parallel and distributed sparsification
- Deep sparsification by effective resistances
- A heuristic for better clustering

Matrix Ordering

- Whenever for all vectors *x* we have

$$x^T G x \leq x^T H x$$

- We write

$$G \preceq H$$

- In this notation, a spectral sparsifier *H* satisfies

$$(1-\epsilon)G \preceq H \preceq (1+\epsilon)G$$

Sums of random matrices

[Tropp '12, adapted]:

1. Let **S** be a nxn PSD matrix and $\mathbf{Y_1, \dots, Y_k}$ be independent random PSD matrices also of size nxn.
2. Let $\mathbf{Y = S + \sum_i Y_i}$
3. Let **Z = E[Y]**
4. Suppose $\mathbf{Y_i} \preceq \mathbf{R \cdot Z}$

S : Combinatorial Sketch

$\mathbf{Y_i}$ **: Edges**

$$
\begin{aligned}
Pr\left[Y \preceq (1-\epsilon)Z\right] &\leq n \cdot exp(-\epsilon^2/2R) \;\; \forall \epsilon \in [0,1] \\
Pr\left[Y \succeq (1+\epsilon)Z\right] &\leq n \cdot exp(-\epsilon^2/3R) \;\; \forall \epsilon \in [0,1]
\end{aligned}
$$

R: should be O(C/log n)

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A simple algorithm

1. Compute a **spanner** S':
2. Let H:= S'
3. For every edge e not in H:

   H:=H + k*e, with probability 1/k

H has O(nlog n) + m/k edges

n: number of vertices
m: number of edges

$$(1/2)Ck\log^2 n \cdot G \preceq H \preceq (3/2)Ck\log^2 n \cdot G$$

and a simple proof

- Suppose the graph G has n vertices and m edges
- For simplicity we let G be unweighted (generalization is easy)

- By definition of spanner, for every edge e of G, there is a path p_e in S ' that joins the two endpoints of e and has length logn.

- Algebraically, if G_e is the Laplacian of edge e:

$$G_e \preceq \log n \cdot p_e \preceq \log n \cdot S'$$

and a simple proof

- Apply Tropp's Theorem on: $G' = G + kS$
- Combinatorial Sketch: $S = kS'$
- Samples: $Y_i \preceq kG_e$

$$G_e \preceq \log n \cdot S \implies$$

$$Y_i \preceq kG_e \preceq \frac{1}{C \log n} kC \log^2 n \cdot S \preceq \frac{1}{C \log n} G'$$

Incremental Sparsification for Solving

H has **O(nlog n) + m/k edges**

$$(1/2)Ck\log^2 n \cdot G \preceq H \preceq (3/2)Ck\log^2 n \cdot G$$

[Spielman and Teng]

If we can construct H with same guarantees but only n+m/k edges then we can solve linear systems on Laplacians in O(mlog^{2}n) time

[K, Miller Peng 10]

Use low-stretch tree instead of spanner. It preserves distances **on average**. Use sampling with skewed probability distribution.

Incremental Sparsification for Solving

[K, Miller Peng 10,11]

If A is a symmetric diagonally dominant matrix then an approximate solution to **Ax= b** can be computed in O(m log n log log n log (1/ϵ)) time, where ϵ is the required precision

Fact:

Approximations to the j first eigenvectors can be computed via solving O(j log n) linear systems

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Parallel and Distributed Sparsification

1. Can we do better than incremental sparsification ?
2. Is there a parallel sparsification algorithm?
3. Is there a distributed sparsification algorithm ?

- Spanners hold the key to questions #2, #3
- There are **very efficient** parallel and distributed algorithms for computing spanners. From this we get parallel and distributed incremental sparsification. This doesn't itself imply parallel solvers.
- The main problem is question #1.

Parallel and Distributed Sparsification

- A better combinatorial sketch:
- **t-bundle spanner:** A collection of graphs $S_1 \ldots . S_t$ such that S_i is a spanner for $G - (S_1 + \ldots . + S_{i-1})$

- A t-bundle spanner can be computed with t sequential calls to a spanner computation algorithm.

- If t is small then the algorithm remains efficient

(polylogarithmic parallel time and distributed rounds)

A simple algorithm

1. Compute $O(\log^4 n)$-bundle spanner S
2. Let H:= S
3. For every edge e not in H:

   H:=H + 2*e, with probability 1/2

H has $O(n\log^5 n) + m/2$ edges

n: number of vertices
m: number of edges

$$(1 - 1/\log n)G \preceq H \preceq (1 + 1/\log n)G$$

A simple algorithm

- H has $O(n\log^5 n) + m/2$ edges

$$(1 - 1/\log n)H \preceq G \preceq (1 + 1/\log n)H$$

- Small size reduction (factor of 2)
- Very tight spectral approximation

- Repeat recursively on H. In $O(\log n)$ rounds we get $O(n\log^6 n)$ edges and a constant spectral approximation.

$$1/2G \preceq H \preceq 3/2G$$

A parallel solver

- This is the best known parallel sparsification routine.
- Improves the total work guarantees of a parallel solver recently described by Peng and Spielman.

- Parallel solver that works in polylogarithmic time and does **$O(m\log^3 n)$** work.

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Spielman-Srivastava: Deep sparsification

- Spielman and Srivastava proved: There is a graph H with $O(n \log n/\epsilon^2)$ edges such that
$$(1-\epsilon)G \preceq H \preceq (1+\epsilon)G$$

- The algorithm is based on sampling edges with probabilities proportional to the **effective resistances** of the edges in the graph

- Proof uses similarly Tropp's theorem

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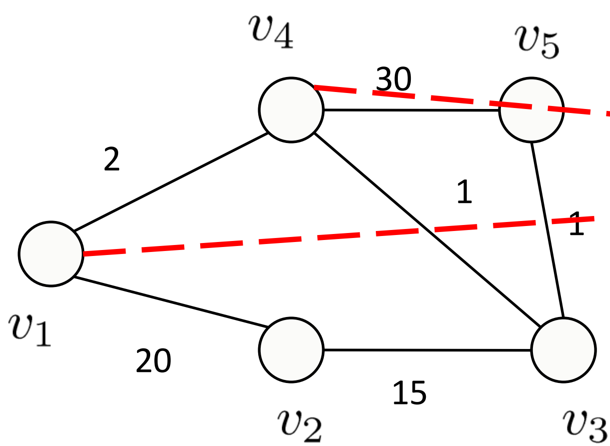
The Incidence Matrix



$$B = W \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

W is the diagonal matrix containing the square roots of edge weights

Spielman-Srivastava: Deep sparsification

- **Effective resistances** can be approximated closely by **solving** O(log n) linear Laplacian systems as follows:

1. Let L be the Laplacian of the graph
2. Let B be the incident matrix of the graph
3. Let Q be a random Johnson-Lindenstrauss projection of size m x O(log n)
4. Solve the systems $L X = B^T Q$
5. Effective resistance between vertices i and j is equal to the $||X_i - X_j||_2$ where X_i is the ith row of X

- The solution X is a n x O(log n) matrix.
- Each row can be interpreted as an **embedding** of corresponding vertex to the O(log n)-dimensional Euclidean space.
- Let's call this the **effective resistance embedding**.

Fast Effective Resistances

An implementation of the Spielman-Srivastava algorithm for the quick computation of **many** effective resistances in an electrical resistive network. Effective resistances are equivalent to **commute times** of the random walk in the corresponding graph.

The code runs in MATLAB. Authored by Richard Garcia Lebron.

Download **Dependence:** CMG solver

- http://ccom.uprrp.edu/~ikoutis/SpectralAlgorithms.htm

Heuristic: clustering based on the effective resistance embedding

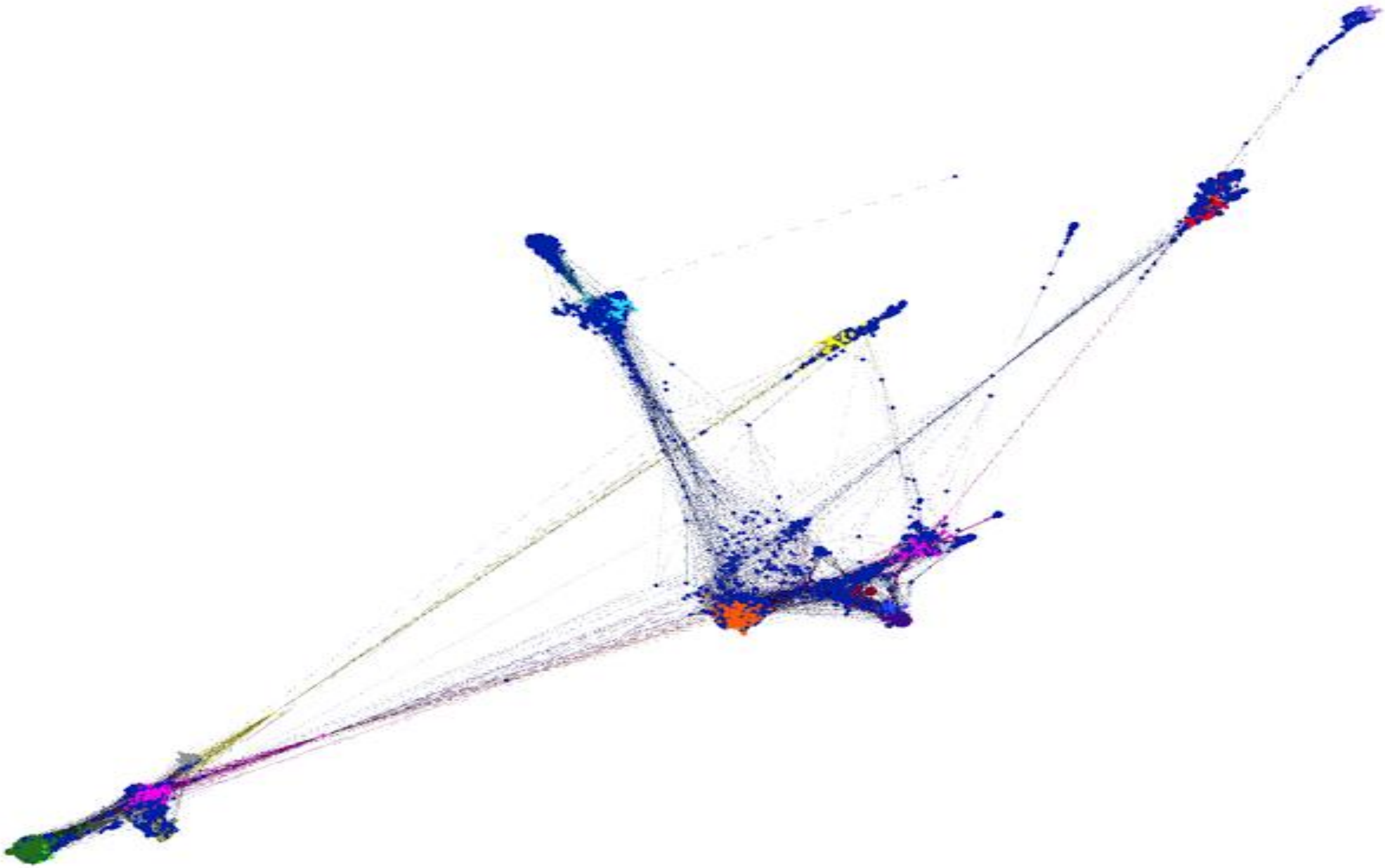
- Small effective resistance for an edge e means that there are a lot of short connections between the two endpoints .

- Points that are close in the geometric embedding should be close in this connectivity sense in the graph.

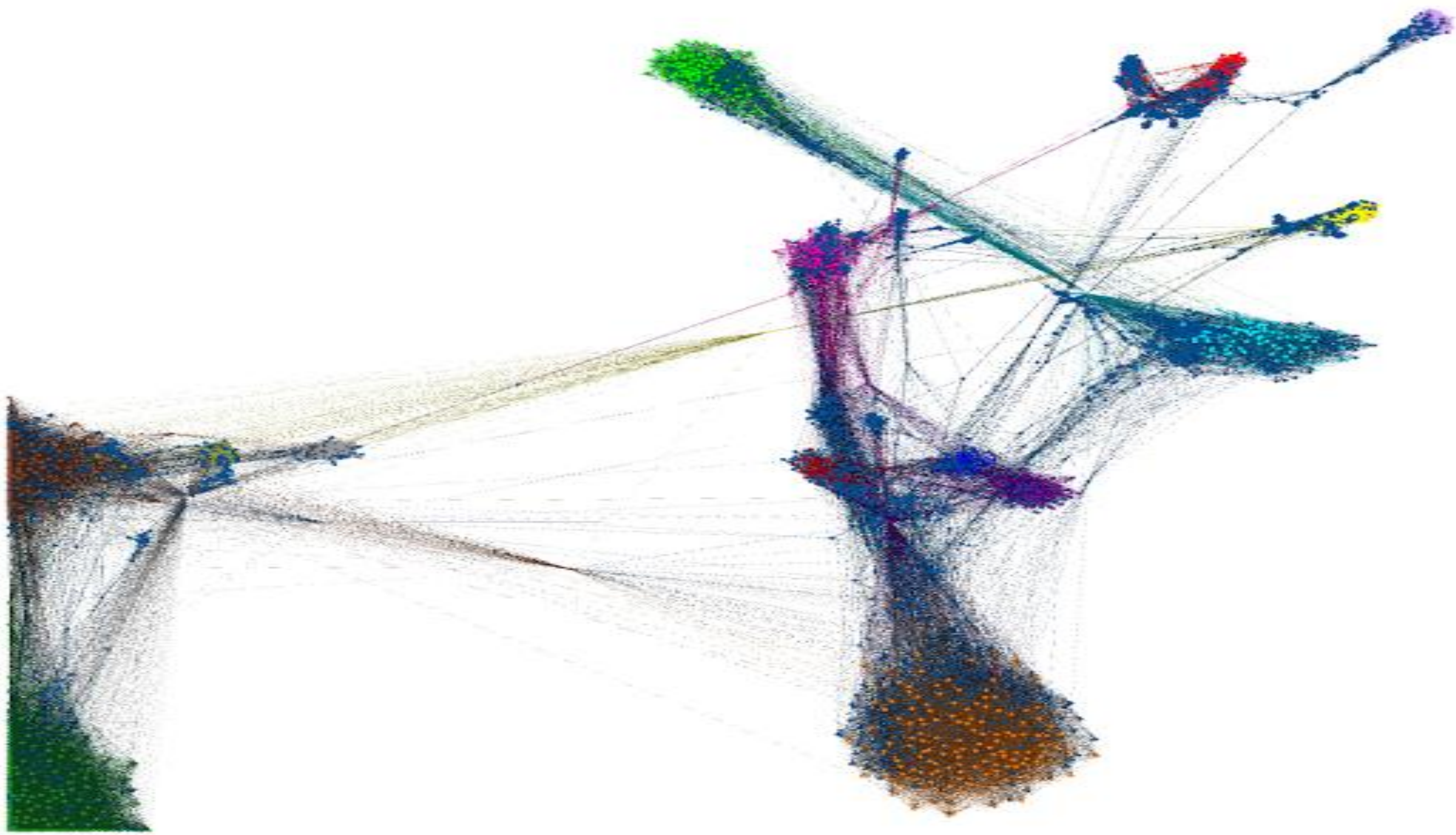
- **Idea:** Produce a k-clustering of the graph by running k-means on the effective resistance embedding

- Produced clusterings appear to have better properties than clusterings based on geometric embeddings using the k-first eigenvectors. It is also much faster for most values of k.

Visualization of unweighted social network graph



Visualization of the same graph using effective resistances as weights



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Spectral Sparsification vs Algebraic Sketching

- The sparsifier H of G has the form

$$H = B^T S^T S B$$

- Here S is a **diagonal matrix** containing the reweight factors of the edges and B is the incidence matrix for G

- Algebraic sketching : instead of solving a regression problem with a matrix B, solve instead one with SB where S is a **sparse projection matrix**
- Goal is

$$(1-\epsilon)B^T B \leq B^T S^T S B \leq (1+\epsilon)B^T B$$

Spectral Sparsification vs Algebraic Sketching

- So, spectral graph sparsification is
  a **special instance** of algebraic sketching

- Algebraic sketching takes a **very** **tall and thin** matrix
  and finds a nearly equivalent **tall and thin** matrix

- In the graph sparsification case graph combinatorics
  allow for a much **tighter control** **on size reduction**